LESSON 7 : MORE FACTORIZATIONS AND MANIPULATIONS

- Previously, you learned Simon's Favorite factoring trick which allowed you complete factorizations such as:
  $xy + x + y + 1 = (x+1)(y+1)$ and $xy - x - y + 1 = (x-1)(y-1)$
- This lesson focuses on algebraic concepts you may be familiar with, but applies them in a very powerful way
- Remember, sometimes we use concepts based in algebra even when there aren't xs or ys in the problem. Keep an open mind!
- For additional practice, see Chapter 7 of the AoPS Volume 1 book

7.1: FACTORIZATIONS

- If asked to evaluate $100002^2 - 99998^2$, calculating it the traditional way would be very tedious. But, recalling the simple factorization of $x^2 - y^2 = (x+y)(x-y)$ simplifies the question greatly. Now we proceed:
  $100002^2 - 99998^2 = (100002 - 99998)(100002 + 99998) = 80000$

- These helpful applications of factorizations aren't confined to these kind of questions. Usually, there are 4 main factorizations that will come into play:
  1. $a^2 - b^2 = (a-b)(a+b)$
  2. $a^2 + b^2 = (a+b)^2 - 2ab$
  3. $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$
  4. $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

- Factorization 2 can sometimes be helpful in factoring expressions further:
  $x^4 + y^4 + x^2y^2 = (x^2+y^2)^2 - 2x^2y^2 + x^2y^2$
  $= (x^2+y^2)^2 - (xy)^2$
  $= (x^2 + y^2 - xy)(x^2 + y^2 + xy)$

- These factorizations are found in problems in varying forms. Sometimes, instead of the sum or difference of the cubes showing up, one of the factors will, and you need to work towards the sum/diff.
- Remember...these manipulations may be obvious when variables are involved, but they become harder to see when the problem is in terms of numbers

- Ex. 1: If $\frac{1}{a+c} = \frac{1}{a} + \frac{1}{c}$, find $(\frac{a}{c})^3$

  Clearing denominators by multiplying each side by $ac(a+c)$ gives:

  $$ac = c(a+c) + a(a+c)$$

  Rearranging,

  $$ac = ac + c^2 + a^2 + ac$$
  $$0 = a^2 + ac + c^2$$

  Now, we see that $a^2 + ac + c^2$ is a factor of $a^3 - c^3$. It's only missing the $(a-c)$ term. Multiplying this on each side:

  $$0 = (a^2 + ac + c^2)(a-c) = a^3 - c^3$$

  This means that $a^3 = c^3$ and $\therefore$ $(\frac{a}{c})^3 = 1$

- Ex. 2: Find the sum

  $$\frac{1}{3+2\sqrt{2}} + \frac{1}{2\sqrt{2}+\sqrt{7}} + \frac{1}{\sqrt{7}+\sqrt{6}} + \frac{1}{\sqrt{6}+\sqrt{5}} + \frac{1}{\sqrt{5}+2} + \frac{1}{2+\sqrt{3}}$$

  First, bring each integer inside of the square root so that everything is written in the same form.

  Notice that each term is in the form $\frac{1}{\sqrt{m}+\sqrt{n}}$, where $m = n+1$.

  Rationalizing the terms of this form by multiplying by $\frac{\sqrt{m}-\sqrt{n}}{\sqrt{m}-\sqrt{n}}$ leaves only $\sqrt{m}-\sqrt{n}$ for each term. Writing this out and canceling the terms, we get:

  $$\sqrt{9} - \sqrt{8} + \sqrt{8} - \sqrt{7} + \sqrt{7} - \sqrt{6} + \sqrt{6} - \sqrt{5} + \sqrt{5} - \sqrt{4} + \sqrt{4} - \sqrt{3}$$
  $$= 3 - \sqrt{3}$$

7.2: MANIPULATIONS

- Manipulations require recognizing and implementing factorizations/expansions within a problem

- Consider the following problem:
  If $x+\frac{1}{x}=1$, find $x^3+\frac{1}{x^3}$.

  One approach would be to solve the first equation for x and plug it into the second one. A smarter approach, however, is the following:

  By cubing both sides, we obtain that $1^3=(x+\frac{1}{x})^3$. Expanding:

  $$1=x^3+3x^2(\frac{1}{x})+3x(\frac{1}{x})^2+\frac{1}{x^3}$$
  $$1=x^3+3x+\frac{3}{x}+\frac{1}{x^3}$$

  Factoring this, we get that $1=x^3+3(x+\frac{1}{x})+\frac{1}{x^3}$. Since we know $x+\frac{1}{x}=1$, plugging in gives us that $1=x^3+3(1)+\frac{1}{x^3}$.

  $\therefore$ $x^3+\frac{1}{x^3}=-2$.

- Problems involving the sum of a term and its reciprocal can often be tackled by raising the initial equation to various powers

- Another type of problem that can be solved by raising equations to various powers is one in which we are given two of the quantities xy, $x+y$, x^2+y^2, x^3+y^3 and asked to solve for a third

- By squaring or cubing $x+y$, we end up with an expression that may have an xy, an $x+y$, an x^2+y^2, or x^3+y^3:

  $$(x+y)^2=x^2+y^2+2xy$$
  $$(x+y)^3=x^3+y^3+3xy(x+y)$$

- You will usually have two of these three values and can determine the third

- <u>Ex. 1:</u> if $a+b=1$ and $a^2+b^2=2$, find a^4+b^4

By first squaring the sum of the variables, we can solve for ab. Note that in these types of questions, we will almost always end up solving for ab somewhere along the way.

$$(a+b)^2 = a^2 + b^2 + 2ab$$
$$1^2 = 2 + 2ab$$
$$\therefore \quad ab = -\frac{1}{2}$$

Now, squaring $a^2 + b^2$ to get $a^4 + b^4$:

$$(a^2+b^2)^2 = a^4 + b^4 + 2(ab)^2$$
$$2^2 = a^4 + b^4 + 2(-\frac{1}{2})^2$$
$$\therefore a^4 + b^4 = \frac{7}{2}$$

Problems:

(Problems pulled from AoPS Vol. 1 unless indicated otherwise - come from various competitions)

1. Find the sum

$$\frac{1}{\sqrt[3]{1}+\sqrt[3]{2}+\sqrt[3]{4}} + \frac{1}{\sqrt[3]{4}+\sqrt[3]{6}+\sqrt[3]{9}} + \frac{1}{\sqrt[3]{9}+\sqrt[3]{12}+\sqrt[3]{16}}$$

2. Find $x^6 + \frac{1}{x^6}$ if $x + \frac{1}{x} = 3$.

3. Find $x^4 + \frac{1}{x^4}$ if $x - \frac{1}{x} = 5$.

4. Given that $9876^2 = 97535376$, find 9877^2.

5. What is the sum of the prime factors of $2^{16} - 1$?

6. If the sum of two numbers is 1 and their product is 1, what is the sum of their cubes?

7. If $x + y = 4$ and $xy = 2$, then find $x^6 + y^6$.

8. Find all possible values of $x^3 + \dfrac{1}{x^3}$ given that $x^2 + \dfrac{1}{x^2} = 7$.

9. If r and s are the roots of $x^2 + px + q = 0$, then find each of the following in terms of p and q: (Remember Vieta's?)

i. $r^2 + s^2$
ii. $r - s$
iii. $r^2s + rs^2$
iv. $r^4 + s^4$

10. Find all possible values of ab given that $a+b=2$ and $a^4+b^4=16$

11. If $a^3-b^3=24$ and $a-b=2$, then find all possible values of $a+b$.

12. (AIME 1986 #4) Determine $3x_4+2x_5$ if x_1, x_2, x_3, x_4 and x_5 satisfy the system of equations below:

$$2x_1+x_2+x_3+x_4+x_5=6$$
$$x_1+2x_2+x_3+x_4+x_5=12$$
$$x_1+x_2+2x_3+x_4+x_5=24$$
$$x_1+x_2+x_3+2x_4+x_5=48$$
$$x_1+x_2+x_3+x_4+2x_5=96$$

13. (Mandelbrot #2) If q is an integer that can be expressed as the sum of two integer squares, show that both $2q$ and $5q$ can be expressed as the sum of two integer squares.

By first squaring the sum of the variables, we can solve for ab. Note that in these types of questions, we will almost always end up solving for ab somewhere along the way.

$$(a+b)^2 = a^2 + b^2 + 2ab$$
$$1^2 = 2 + 2ab$$
$$\therefore\ ab = -\frac{1}{2}$$

Now, squaring $a^2 + b^2$ to get $a^4 + b^4$:

$$(a^2+b^2)^2 = a^4 + b^4 + 2(ab)^2$$
$$2^2 = a^4 + b^4 + 2(-\frac{1}{2})^2$$
$$\therefore a^4 + b^4 = \frac{7}{2}$$

Problems: (steps labeled 1 or 2 so you can follow work easier)

(Problems pulled from AoPS Vol. 1 unless indicated otherwise - come from various competitions)

$$\frac{1}{\sqrt[3]{1}+\sqrt[3]{2}+\sqrt[3]{4}} \cdot \frac{\sqrt[3]{2}-\sqrt[3]{1}}{\sqrt[3]{2}-\sqrt[3]{1}} = \frac{\sqrt[3]{2}-\sqrt[3]{1}}{2-1} = \sqrt[3]{2}-\sqrt[3]{1}$$

1. Find the sum

$$\frac{1}{\sqrt[3]{1}+\sqrt[3]{2}+\sqrt[3]{4}} + \frac{1}{\sqrt[3]{4}+\sqrt[3]{6}+\sqrt[3]{9}} + \frac{1}{\sqrt[3]{9}+\sqrt[3]{12}+\sqrt[3]{16}}$$

factor of 3-2

factor 4-3

1) consider the following: $(\sqrt[3]{2})^3 - (\sqrt[3]{1})^3 = (\sqrt[3]{2}-\sqrt[3]{1})(\sqrt[3]{2^2}+\sqrt[3]{2}\cdot\sqrt[3]{1}+\sqrt[3]{1^2})$
$= (\sqrt[3]{2}-\sqrt[3]{1})(\sqrt[3]{4}+\sqrt[3]{2}+\sqrt[3]{1}) \Rightarrow$ what we have i[s] a factor of 2- when we use dif of cubes to fact[or]

2) So, we end up with:

$$\cancel{\sqrt[3]{2}}-\sqrt[3]{1}+\cancel{\sqrt[3]{3}}-\cancel{\sqrt[3]{2}}+\sqrt[3]{4}-\cancel{\sqrt[3]{3}} = \boxed{\sqrt[3]{4}-1}$$

2. Find $x^6 + \frac{1}{x^6}$ if $x + \frac{1}{x} = 3$.

1) $(x+\frac{1}{x})^2 = 9 = x^2 + \frac{1}{x^2} + 2 \Rightarrow x^2 + \frac{1}{x^2} = 7$

2) $(x^2+\frac{1}{x^2})^3 = 7^3 = x^6 + \frac{1}{x^6} + 3(x^2+\frac{1}{x^2})$

$343 = x^6 + \frac{1}{x^6} + 3(7) \Rightarrow \boxed{x^6 + \frac{1}{x^6} = 322}$

3. Find $x^4 + \frac{1}{x^4}$ if $x - \frac{1}{x} = 5$.

1) $(x-\frac{1}{x})^2 = x^2 + \frac{1}{x^2} - 2 = 5^2$

$\Rightarrow x^2 + \frac{1}{x^2} = 27$

2) $(x^2+\frac{1}{x^2})^2 = x^4 + \frac{1}{x^4} + 2 = 27^2$

$\boxed{x^4 + \frac{1}{x^4} = 727}$

4. Given that $9876^2 = 97535376$, find 9877^2.

$9877^2 - 9876^2 = (9877-9876)(9877+9876)$

$9877^2 = (1)(19753) + 9876^2 = 19753 + 97535376$

$= \boxed{97555129}$

$$\begin{array}{r} 9877 \\ 9876 \\ \hline 19753 \end{array} \qquad \begin{array}{r} 97535376 \\ 19753 \\ \hline 97555129 \end{array}$$

5. What is the sum of the prime factors of $2^{16} - 1$?

Factor until you can't anymore!

$2^{16} - 1$

$= (2^8 - 1)(2^8 + 1)$

$= (2^4 - 1)(2^4 + 1)(2^8 + 1)$

$= (2^2 - 1)(2^2 + 1)(2^4 + 1)(2^8 + 1) = (3)(5)(17)(257)$

$$\begin{array}{r} 257 \\ 17 \\ 5 \\ 3 \\ \hline \boxed{282} \end{array}$$

6. If the sum of two numbers is 1 and their product is 1, what is the sum of their cubes?

$x + y = 1 \qquad xy = 1$

$(x+y)^3 = x^3 + y^3 + 3xy(x+y)$

$1^3 = x^3 + y^3 + 3(1)(1)$

$\boxed{x^3 + y^3 = -2}$

7. If $x + y = 4$ and $xy = 2$, then find $x^6 + y^6$.

1) $(x+y)^2 = x^2 + y^2 + 2xy$

$4^2 = x^2 + y^2 + 2(2)$

$x^2 + y^2 = 12$

2) $(x^2+y^2)^3 = x^6 + y^6 + 3x^2y^2(x^2+y^2)$

$12^3 = x^6 + y^6 + 3(xy)^2(x^2+y^2)$

$12^3 = x^6 + y^6 + 3(2)^2(12)$

$12^3 - 3(2)^2(12) = \boxed{x^6 + y^6 = 1584}$

8. Find all possible values of $x^3 + \dfrac{1}{x^3}$ given that $x^2 + \dfrac{1}{x^2} = 7$.

$(x + \frac{1}{x})^2 = \underbrace{x^2 + \frac{1}{x^2}}_{} + 2$

$7 + 2 = 9$

$\Rightarrow x + \frac{1}{x} = \pm 3$

for $x + \frac{1}{x} = 3$: $(x + \frac{1}{x})^3 = x^3 + \frac{1}{x^3} + 3(x + \frac{1}{x})$

$3^3 = x^3 + \frac{1}{x^3} + 3(3)$

$x^3 + \frac{1}{x^3} = 18$

for $x + \frac{1}{x} = -3$: $(-3)^3 = x^3 + \frac{1}{x^3} + (3)(-3)$

$-27 + 9 = x^3 + \frac{1}{x^3} = -18$

$\therefore \boxed{x^3 + \frac{1}{x^3} = \pm 18}$

9. If r and s are the roots of $x^2 + px + q = 0$, then find each of the following in terms of p and q: (Remember Vieta's?) sum of roots $= -b/a = -p$

prod. of roots $= c/a = q$

i. $r^2 + s^2 = (r+s)^2 - 2rs = \boxed{p^2 - 2q}$

ii. $r - s = \boxed{\pm\sqrt{p^2 - 4q}}$

iii. $r^2s + rs^2 = rs(r+s) = \boxed{-pq}$

iv. $r^4 + s^4 = (r^2+s^2)^2 - 2r^2s^2 = (r^2+s^2)^2 - 2(rs)^2$

$= (p^2 - 2q)^2 - 2(q)^2$

$= p^4 - 4p^2q + 4q^2 - 2q^2 = \boxed{p^4 - 4p^2q + 2q^2}$

$(r-s)^2 = r^2 + s^2 - 2rs$

$= p^2 - 2q - 2q$

$= p^2 - 4q$

$\Rightarrow r - s = \pm\sqrt{p^2 - 4q}$

10. Find all possible values of ab given that $a+b=2$ and $a^4+b^4=16$

1) $(a+b)^2 = a^2+b^2+2ab$
$2^2 = a^2+b^2+2ab$
$a^2+b^2 = 4-2ab$

2) $(a^2+b^2)^2 = \underbrace{a^4+b^4}_{16} + 2a^2b^2$
$(4-2ab)^2 = 16 + 2(ab)^2$
Let $ab=y$: $(4-2y)^2 = 16+2y^2$
$\cancel{16} - 16y + 4y^2 = \cancel{16} + 2y^2$
$2y^2 - 16y = 0$
$2y(y-8) = 0$
$y = 0, 8$ $\therefore$ $\boxed{ab = 0 \text{ or } ab = 8}$

11. If $a^3-b^3=24$ and $a-b=2$, then find all possible values of $a+b$.

1) $(a-b)^3 = a^3-b^3-3ab(a-b)$
$2^3 = 24 - 3ab(2)$
$6ab = 16$
$ab = \frac{16}{6} = \frac{8}{3}$

2) $a^3-b^3 = (a-b)(a^2+ab+b^2)$
$a^2+ab+b^2 = 12$
$(a+b)^2 - ab = 12$
$(a+b)^2 = 12 + ab = 12 + \frac{8}{3} = \frac{44}{3}$
$a+b = \pm\sqrt{\frac{44}{3}} = \boxed{\pm 2\sqrt{\frac{11}{3}}}$

12. (AIME 1986 #4) Determine $3x_4+2x_5$ if x_1, x_2, x_3, x_4 and x_5 satisfy the system of equations below:

$$2x_1+x_2+x_3+x_4+x_5=6$$
$$x_1+2x_2+x_3+x_4+x_5=12$$
$$x_1+x_2+2x_3+x_4+x_5=24$$
$$x_1+x_2+x_3+2x_4+x_5=48$$
$$x_1+x_2+x_3+x_4+2x_5=96$$

+

$6(x_1+x_2+x_3+x_4+x_5) = 186$
$\Rightarrow x_1+x_2+x_3+x_4+x_5 = 31$

$\cancel{x_1}+\cancel{x_2}+\cancel{x_3}+2x_4+x_5 = 48$
$-(\cancel{x_1}+\cancel{x_2}+\cancel{x_3}+x_4+x_5) = -31$
$x_4 = 17$

$\cancel{x_1}+\cancel{x_2}+\cancel{x_3}+\cancel{x_4}+2x_5 = 96$
$-(\cancel{x_1}+\cancel{x_2}+\cancel{x_3}+\cancel{x_4}+x_5) = -31$
$x_5 = 65$

$\therefore 3x_4+2x_5 = 3(17)+2(65)$
$= \boxed{181}$

13. (Mandelbrot #2) If q is an integer that can be expressed as the sum of two integer squares, show that both $2q$ and $5q$ can be expressed as the sum of two integer squares.

$q = m^2+n^2$, where m and n are integers

$2q = 2m^2+2n^2$, now we must show that $2m^2$ and $2n^2$ can be achieved by a sum of squares. I see m^2 and n^2, but no mn term. $\therefore$ I'll try $(m+n)^2$ and $(m-n)^2$

$(m+n)^2+(m-n)^2 = \boxed{2m^2+2n^2}$ ✓

$5q = 5m^2+5n^2$, once again, there is m^2 and n^2, but no mn terms. so, we need one $(-)$ term and one $(+)$ term. Also, since the leading coefficient is 5, I'll try: $(2m-n), (m+2n)$

$(2m-n)^2+(m+2n)^2$
$= 4m^2 - \cancel{4mn} + n^2 + m^2 + \cancel{4mn} + 4n^2$
$= \boxed{5m^2+5n^2}$ ✓

$\boxed{\therefore 2q \text{ can be written as } (m+n)^2+(m-n)^2 \text{ and } 5q \text{ as } (2m-n)^2+(m+2n)^2.}$