Lesson 4: Algebraic Manipulations

(Factoring is Factoring and Simon’s Favorite Factoring Trick)

Factoring is Factoring is a technique that lets us solve algebraic diophantine equations (equations with integer solutions). It lets us factor one side of an expression algebraically and the other side numerically.

Simon's Favorite Factoring Trick is a technique that is used to factor equations that don't necessarily have an easy factorization. Used in conjunction with Factoring is Factoring, it lets us find integer solutions to equations involving multiple variables.

In order to understand these, we will go through the thought process of thinking through problems which involve it.

Factoring is Factoring

We begin with a simple problem:

- Find all **positive integer** solution pairs (x, y) such that $xy = 12$. (x, y) such that $xy = 12$.

Note the words “positive integers”. If these words weren't there, there would be infinitely many solutions. Some solutions might be $(3,4), (\pi, \frac{12}{\pi}), (.0001, 120000), \cdots$

We solve this by just listing out all the factor pairs of 12.

x	y
1	12
2	6
3	4
4	3
6	2
12	1

So the solutions are $\boxed{(1,12),(2,6),(3,4),(4,3),(6,2),(12,1)}$.

What about if we remove the word **positive**? Are the solutions still the same?

No, because now we have to consider negative solutions as well. The solutions now also include .
$(-1,-12),(-2,-6),(-3,-4),(-4,-3),(-6,-2),(-12,-1)$.

Now we try to extend this problem a bit:

- Find all **positive integer** solutions (x,y) to $x(y+1)=15$.

Again, note the words **positive integer**.

To solve this, we again look at factor pairs of 15.

x	$y+1$	y
1	15	14
3	5	4
5	3	2
~~15~~	~~1~~	~~0~~

So are our solutions just $(1,15),(3,5),(5,3),(15,1)$?

No, because those are solutions for $(x,y+1)$. We are looking for solutions to (x,y), so we must subtract 1 from $y+1$ to find y.

Okay, so does that mean our solutions are just $(1,14),(3,4),(5,2),(15,0)$?

Again, we are missing something. Look back at the constraints in the problem. What are the bolded words? **"positive integer"**

One of our solutions doesn't fit this constraint, specifically $(15,0)$, since 0 is not positive. So now our solutions are $\boxed{(1,14),(3,4),(5,2)}$.

- Find $x+y$ if x and y are **positive integer** solutions to $xy^2+xy+2y+2=14$.

Notice we only asked for the value of $x+y$, not to list all solutions. This hints that there is only one solution that fits all constraints.

First of all, we see if we can factor the first expression. Using the basic factoring by grouping method, we can factor the LHS to $(xy+2)(y+1)$.

Now, we go ahead and make another factor table. This time, our factors are $xy+2$ and $y+1$. We then have some additional columns (y, xy, and x) to help us solve for x and y.

$xy+2$	$y+1$	y	xy	x
1	14	13	-1	-1/13
2	7	6	0	0
7	2	**1**	5	**5**
14	1	0	12	undefined

We notice that there is only one row that fits our constraints of x and y being positive integers, so $x+y=6$. x and y being positive integers, so $x+y=6$.

Some common tips to use with Factoring is Factoring:

- Move everything algebraic to one side, and everything numeric to the other side
- Use your number sense formulas to make sure you are not missing any factor pairs:
    - If $n=2^3\cdot 3^2\cdot 5^7$, n has $(3+1)(2+1)(7+1)=96$ factors
- Make sure you factored your expression correctly. A single incorrect sign often throws you off completely.
- Ensure that your solutions all fit the constraints of the problem. Look for the words **positive, non-negative** especially.
- Don't try to use Factoring is Factoring unless your solutions are all integers.

Simon's Favorite Factoring Trick

Simon's Favorite Factoring Trick is named after an old user on the AoPS forums that popularized this technique. It lets us perform Factoring is Factoring on non factorable expressions.

- Find all non-negative integer solutions (a, b) to $ab + a + b = 3$.

With Factoring is Factoring, we look to factor the algebraic side of this expression. However, it seems that $ab + a + b$ is unfactorable.

It does look slightly familiar, however, and after some experimentation, we see that $\underline{ab + a + b} + 1 = (a+1)(b+1)$. So in order to make the equation solvable via Factoring is Factoring, we add 1 to both sides of the equation, so that our LHS looks like something we already know how to factor.

Thus, we have reduced this problem into something we already know how to solve. The problem essentially becomes

- Find all non-negative integer solutions (a, b) to $(a+1)(b+1) = 4$.

Of course, we just make a factor table and list out possible solutions now:

$a+1$	$b+1$	a	b
1	4	0	3
2	2	1	1
4	1	3	0

Now we see our solutions are $\boxed{(0,3),(1,1),(3,0)}$.

Some common factorizations used by SFFT:

$(a+1)(b+1) = ab + a + b + 1$

$(a-1)(b-1) = ab - a - b + 1$

$(a+1)(b-1) = ab - a + b - 1$

Challenge problems

1. Find the number of solutions (a, b, c, d, e, f, g) to
   $abc = 70$
   $cde = 71$
   $efg = 72$

   (A) 60
   (B) 72
   (C) 84
   (D) 96
   (E) 100

2. Let $a, m, c > 0$ and a, m, c are coprime. Find $a^2 + m^2 + c^2$ if $amc + am + mc + ac + a + m + c = 29$. $a, m, c > 0$ and a, m, c are coprime. Find $a^2 + m^2 + c^2$ if $amc + am + mc + ac + a + m + c = 29$.
   (A) 15
   (B) 17
   (C) 19
   (D) 21
   (E) 23

3. Two different prime numbers between 4 and 18 are chosen. When their sum is subtracted from their product, which of the following numbers could be obtained? 4 and 18 are chosen. When their sum is subtracted from their product, which of the following numbers could be obtained?
   (A) 21
   (B) 60
   (C) 119
   (D) 180
   (E) 231

4. Find $3x^2y^2$ if x and y are integers such that $y^2 + 3x^2y^2 = 30x^2 + 517$.

5. A rectangular floor measures a by b feet, where a and b are positive integers with $b > a$. An artist paints a rectangle on the floor with the sides of the rectangle parallel to the sides of the floor. The unpainted part of the floor forms a border of width 1 foot around the painted rectangle and occupies half of the area of the entire floor. How many possibilities are there for the ordered pair (a, b)?