Advanced AMC Problemset 1

Pulled from 2012 AMC 12A and 2011 AMC 12A

Problem 9

A year is a leap year if and only if the year number is divisible by 400 (such as 2000) or is divisible by 4 but not by 100 (such as 2012). The 200th anniversary of the birth of novelist Charles Dickens was celebrated on February 7, 2012, a Tuesday. On what day of the week was Dickens born?

$\textbf{(A)}\ \text{Friday} \qquad \textbf{(B)}\ \text{Saturday} \qquad \textbf{(C)}\ \text{Sunday} \qquad \textbf{(D)}\ \text{Monday} \qquad \textbf{(E)}\ \text{Tuesday}$

Problem 10

A triangle has area 30, one side of length 10, and the median to that side of length 9. Let θ be the acute angle formed by that side and the median. What is $\sin\theta$?

$\textbf{(A)}\ \frac{3}{10} \qquad \textbf{(B)}\ \frac{1}{3} \qquad \textbf{(C)}\ \frac{9}{20} \qquad \textbf{(D)}\ \frac{2}{3} \qquad \textbf{(E)}\ \frac{9}{10}$

Problem 11

Alex, Mel, and Chelsea play a game that has 6 rounds. In each round there is a single winner, and the outcomes of the rounds are independent. For each round the probability that Alex wins is $\frac{1}{2}$, and Mel is twice as likely to win as Chelsea. What is the probability that Alex wins three rounds, Mel wins two rounds, and Chelsea wins one round?

$\textbf{(A)}\ \frac{5}{72} \qquad \textbf{(B)}\ \frac{5}{36} \qquad \textbf{(C)}\ \frac{1}{6} \qquad \textbf{(D)}\ \frac{1}{3} \qquad \textbf{(E)}\ 1$

Problem 12

A power boat and a raft both left dock A on a river and headed downstream. The raft drifted at the speed of the river current. The power boat maintained a constant speed with respect to the river. The power boat reached dock B downriver, then immediately turned and traveled back upriver. It eventually met the raft on the river 9 hours after leaving dock A. How many hours did it take the power boat to go from A to B?

$\textbf{(A)}\ 3 \qquad \textbf{(B)}\ 3.5 \qquad \textbf{(C)}\ 4 \qquad \textbf{(D)}\ 4.5 \qquad \textbf{(E)}\ 5$

Problem 13

Paula the painter and her two helpers each paint at constant, but different, rates. They always start at $8{:}00\ \text{AM}$, and all three always take the same amount of time to eat lunch. On Monday the three of them painted 50% of a house, quitting at $4{:}00\ \text{PM}$. On Tuesday, when Paula wasn't there, the two helpers painted only 24% of the house and quit at $2{:}12\ \text{PM}$. On Wednesday Paula worked by herself and finished the house by working until $7{:}12\ \text{PM}$. How long, in minutes, was each day's lunch break?

$\textbf{(A)}\ 30 \qquad \textbf{(B)}\ 36 \qquad \textbf{(C)}\ 42 \qquad \textbf{(D)}\ 48 \qquad \textbf{(E)}\ 60$

Problem 14

Suppose a and b are single-digit positive integers chosen independently and at random. What is the probability that the point (a,b) lies above the parabola $y=ax^2-bx$?

$\textbf{(A) } \frac{11}{81} \qquad \textbf{(B) } \frac{13}{81} \qquad \textbf{(C) } \frac{5}{27} \qquad \textbf{(D) } \frac{17}{81} \qquad \textbf{(E) } \frac{19}{81}$

Problem 15

A 3×3 square is partitioned into 9 unit squares. Each unit square is painted either white or black with each color being equally likely, chosen independently and at random. The square is then rotated 90° clockwise about its center, and every white square in a position formerly occupied by a black square is painted black. The colors of all other squares are left unchanged. What is the probability that the grid is now entirely black?

$\textbf{(A) } \dfrac{49}{512} \qquad \textbf{(B) } \dfrac{7}{64} \qquad \textbf{(C) } \dfrac{121}{1024} \qquad \textbf{(D) } \dfrac{81}{512} \qquad \textbf{(E) } \dfrac{9}{32}$

Problem 16

Circle C_1 has its center O lying on circle C_2. The two circles meet at X and Y. Point Z in the exterior of C_1 lies on circle C_2 and $XZ=13$, $OZ=11$, and $YZ=7$. What is the radius of circle C_1?

$\textbf{(A) } 5 \qquad \textbf{(B) } \sqrt{26} \qquad \textbf{(C) } 3\sqrt{3} \qquad \textbf{(D) } 2\sqrt{7} \qquad \textbf{(E) } \sqrt{30}$

Problem 16

Each vertex of convex polygon $ABCDE$ is to be assigned a color. There are 6 colors to choose from, and the ends of each diagonal must have different colors. How many different colorings are possible?

$\textbf{(A) } 2520 \qquad \textbf{(B) } 2880 \qquad \textbf{(C) } 3120 \qquad \textbf{(D) } 3250 \qquad \textbf{(E) } 3750$

Problem 17

Let S be a subset of $\{1,2,3,\dots,30\}$ with the property that no pair of distinct elements in S has a sum divisible by 5. What is the largest possible size of S?

$\textbf{(A) } 10 \qquad \textbf{(B) } 13 \qquad \textbf{(C) } 15 \qquad \textbf{(D) } 16 \qquad \textbf{(E) } 18$

Problem 17

Circles with radii 1, 2, and 3 are mutually externally tangent. What is the area of the triangle determined by the points of tangency?

$\textbf{(A) } \frac{3}{5} \qquad \textbf{(B) } \frac{4}{5} \qquad \textbf{(C) } 1 \qquad \textbf{(D) } \frac{6}{5} \qquad \textbf{(E) } \frac{4}{3}$

Problem 19

Adam, Benin, Chiang, Deshawn, Esther, and Fiona have internet accounts. Some, but not all, of them are internet friends with each other, and none of them has an internet friend outside this group. Each of them has the same number of internet friends. In how many different ways can this happen?

$\textbf{(A)}\ 60 \qquad \textbf{(B)}\ 170 \qquad \textbf{(C)}\ 290 \qquad \textbf{(D)}\ 320 \qquad \textbf{(E)}\ 660$

Problem 20

Consider the polynomial

$P(x)=\prod_{k=0}^{10}(x^{2^k}+2^k)=(x+1)(x^2+2)(x^4+4)\cdots (x^{1024}+1024)$

The coefficient of x^{2012} is equal to 2^a. What is a?

$\textbf{(A)}\ 5 \qquad \textbf{(B)}\ 6 \qquad \textbf{(C)}\ 7 \qquad \textbf{(D)}\ 10 \qquad \textbf{(E)}\ 24$

Problem 21

Let a, b, and c be positive integers with $a\ge b\ge c$ such that

$$a^2-b^2-c^2+ab=2011\text{ and}$$
$$a^2+3b^2+3c^2-3ab-2ac-2bc=-1997$$

What is a?

$\textbf{(A)}\ 249 \qquad \textbf{(B)}\ 250 \qquad \textbf{(C)}\ 251 \qquad \textbf{(D)}\ 252 \qquad \textbf{(E)}\ 253$

Problem 21

Let $f_1(x)=\sqrt{1-x}$, and for integers $n\ge 2$, let $f_n(x)=f_{n-1}(\sqrt{n^2-x})$. If N is the largest value of n for which the domain of f_n is nonempty, the domain of f_N is $[c]$. What is $N+c$?

$\textbf{(A)}\ -226 \qquad \textbf{(B)}\ -144 \qquad \textbf{(C)}\ -20 \qquad \textbf{(D)}\ 20 \qquad \textbf{(E)}\ 144$